MAXWELL AND DIRAC THEORIES AS AN ALREADY UNIFIED THEORY

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In this paper we formulate Maxwell and Dirac theories as an already unified theory (in the sense of Misner and Wheeler). We introduce Dirac spinors as "Dirac square root" of the Faraday bivector, and use this in order to find a spinorial representation of Maxwell equations. Then we show that under certain circumstances this spinor equation reduces to an equation formally identical to Dirac equation. Finally we discuss certain conditions under which this equation can be really interpreted as Dirac equation, and some other possible interpretations of this result.

I. INTRODUCTION

One very beautiful formal development concerning the structure of the physical theories was given by the presentation of Misner and Wheeler [MW57] of Maxwell and Einstein theories as an already unified theory. Misner and Wheeler were able to describe the electromagnetic field as a kind of square root (the "Maxwell square root" in their terminology) of the Ricci tensor, giving therefore a geometrical description of electromagnetism by means of some algebraic conditions already discovered by Rainich [Ra25]. This was the starting pointing of Wheeler's program called geometrodynamics – see, for example [Mi87].

In this paper we shall consider a somewhat anologous construction for the case of Maxwell and Dirac theories, that is, we try to formulate electromagnetism and (relativistic) quantum mechanics as an already unified theory. Using a terminology analogous to Misner and Wheeler, we shall introduce a Dirac spinor as a kind of square root – the "Dirac square root" – of the electromagnetic field described by the Faraday bivector. Moreover, we use this result in order to show that Maxwell equations can be put in a form which is identical to Dirac equation.

First of all, before addresing our specific problem, we shall introduce the mathematical background used in this paper (sec.2). In particular, we shall use in this paper the Clifford algebra formalism and the concept of Dirac-Hestenes spinor. One can ask if there is a necessity of doing this, if our calculations cannot be performed with usual mathematical tools, etc. In this way we observe two things: firstly, most of the original results of Misner and Wheeler were obtained due to powerful mathematical tools not widely known to physicists at that time (1957), and, secondly, that Clifford algebras seems to us to be a natural framework for physics, and in particular for our problem; calculations are much more easy to be done using Clifford algebras than with the traditional spinor and tensor calculus – an example of how powerful and natural they are is that the use of the Clifford algebra of spacetime enable us to write the Maxwell's eight scalar equations into a single equation [He66]. Moreover, our approach has a clear connection with quaternionic quantum mechanics [Ad95,LR94], and in this way we also clarify some results of [LR94]. Anyway, our approach will be rather pedestrian, and one can easily find rules for translating our results in terms of more usual concepts.

In sec.3 we introduce the Dirac-Hestenes spinor field as a "Dirac square root" of the Faraday bivector field – this is possible once we use a theorem of Rainich, Misner and Wheeler [MW57,Ra25]. Then we shall look for a spinorial representation of Maxwell equations. There are in the literature several different spinorial representations of Maxwell equations. These spinorial representations, as a rule, use as components of the spinor field the components of the electromagnetic field. The deficiency of this representations is obvious: the action of the Lorentz group on the spinor field does not give the correct transformation laws for the electric and magnetic fields – in fact, a spinor changes sign after a 2π rotation, while the electromagnetic field components does not. A detailed study of these kind of spinorial representation of Maxwell equations can be found in [RO90]. The spinorial representation of Maxwell equations we shall use is based on the concept of Dirac-Hestenes spinor, and does not suffer the deficiency indicated above since the Faraday bivector is the "Dirac square" of the Dirac-Hestenes spinor.

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Once we find a spinorial representation of Maxwell equations we study it in details (sec.4), and we show that it can be reduced to a form which is identical to Dirac equation; then we study certain conditions under which this equation can be really interpreted as Dirac equation. Other possible interpretations are also discussed (sec.5), in particular those which may be relevant to approaches to quantum mechanics based on the concept of quantum potential.

II. MATHEMATICAL PRELIMINARIES: CLIFFORD ALGEBRAS AND DIRAC-HESTENES SPINORS

In this section we shall introduce the algebraic structures to be used, and the concept of Dirac-Hestenes spinors in a pedestrian way. We shall work with multivectors instead of multiforms, but the translation is completely trivial for those used to work with the latter.

A. Exterior, Grassmann and Clifford Algebras

Let V be a vector space of dimension n endowed with an interior product $g : V \times V \to I\!R$. Let $\wedge$ be the exterior (or wedge or Grassmann) product, that is, an associative, bilinear and skew-symmetric product of vectors: $(a \wedge b) \wedge c = a \wedge (b \wedge c)$, $(a + \alpha b) \wedge c = a \wedge c + \alpha b \wedge c$, $a \wedge b = -b \wedge a$, $(\forall a, b, c \in V)$. If $\{e_1, \cdots, e_n\}$ is a basis for V, then $\{e_1 \wedge e_2, \cdots, e_1 \wedge e_n; e_2 \wedge e_3, \cdots, e_{n-1} \wedge e_n\}$ is a basis for the vector space $\bigwedge^2(V)$ whose elements are called bivectors (2-vectors). In this way $\wedge : V \times V \to \bigwedge^2(V)$. We can naturally extend the definition of the exterior product for vectors and bivectors, giving trivectors (3-vectors), and so on. We denote by $\bigwedge^k(V)$ $(0 \leq k \leq n)$ the vector space of k-vectors, which is of dimension $\binom{n}{k}$ (we adopt the convention $\bigwedge^0(V) = I\!R$ and $\bigwedge^1(V) = V$). We have $\wedge : \bigwedge^k(V) \times \bigwedge^l(V) \to \bigwedge^{k+l}(V)$, and if $A_k \in \bigwedge^k(V)$ and $A_l \in \bigwedge^l(V)$ then $A_k \wedge B_l = (-1)^{kl} B_l \wedge A_k$. A k-vector is said to be simple if it is the external product of k 1-vectors. Consider the direct sum $\oplus_{k=0}^n \bigwedge^k(V) = \bigwedge(V)$. An element of $\bigwedge(V)$ is called a multivector, and $(\bigwedge(V), \wedge)$ is called exterior algebra. Let $\langle \ \rangle_k$ denote the projector $\langle \ \rangle_k : \bigwedge(V) \to \bigwedge^k(V)$. If $\langle A \rangle_k = A_k$ the multivector A is said to be homogeneous, and k is its grade.

We can extend g to $\bigwedge(V)$. First, define the extension of g to $\bigwedge^k(V)$ as

$$g_k(a_1 \wedge \cdots \wedge a_k, b_1 \wedge \cdots \wedge b_k) = \det\{g(a_i, b_j)\} \tag{1}$$

for simple k-vectors, and then extend to all $\bigwedge^k(V)$ by linearity (by g_0 we mean ordinary multiplication of scalars). We denote by G the extension of g to $\bigwedge(V)$ given by $G = \sum_{k=0}^n g_k$, with $G(A_k, B_l) = 0$ when $k \neq l$. $(\bigwedge(V), \wedge, G)$ is called Grassmann algebra. The structure of Grassmann algebra is obviously richer than the one of exterior algebra. In order to see this, let us first introduce two involutions. The first, called reversion, and denoted by a tilde, is defined by $\tilde{A}_k = (-1)^{k(k-1)/2} A_k$; the name reversion is due to the fact that it reverses the order of the exterior product of the vectors in a simple k-vector, i.e., $(a_1 \wedge \cdots \wedge a_k)\tilde{} = a_k \wedge \cdots \wedge a_1 = (-1)^{k(k-1)/2}(a_1 \wedge \cdots \wedge a_k)$. The second one, called graded involution, and denoted by a hat, is defined by $\hat{A}_k = (-1)^k A_k$. Now, let us define the contraction. The left contraction $\lrcorner$ is defined by [Lo93]

$$G(A \lrcorner B, C) = G(B, \tilde{A} \wedge C), \quad \forall C \in \bigwedge(V), \tag{2}$$

and the right contraction $\llcorner$ is defined by

$$G(A \llcorner B, C) = G(A, C \wedge \tilde{B}), \quad \forall C \in \bigwedge(V). \tag{3}$$

Left and right contractions are related by

$$A_r \lrcorner B_s = (-1)^{r(s-r)} B_s \llcorner A_r, \quad (s > r), \tag{4}$$

and satisfies the following properties:

$$a \lrcorner b = g(a, b), \tag{5}$$

$$a \lrcorner (B \wedge C) = (a \lrcorner B) \wedge C + \hat{B} \wedge (a \lrcorner C), \tag{6}$$

$$a \lrcorner (b \lrcorner C) = (a \wedge b) \lrcorner C, \tag{7}$$

where $a, b \in V$, $B, C \in \bigwedge(V)$. In what follows, whenever there is no danger of confusion, we denote the contraction by a dot : $a \cdot B = a \lrcorner B$ (the dot must not be confused with internal product in spite that for vectors we have eq.(5)).

Let us introduce the Clifford (or geometrical) product $\vee$; given $a \in V$ and $B \in \bigwedge(V)$ we define $a \vee B$ by

$$a \vee B = a \lrcorner B + a \wedge B, \tag{8}$$

and extend this definition to all $\bigwedge(V)$ by associativity. $(\bigwedge(V), \wedge, G, \vee)$ is called a Clifford algebra (denoted by $Cl(V, g)$). In order to simplify the notation, we denote the Clifford product simply by justaposition, i.e.:

$$aB = a \cdot B + a \wedge B. \tag{9}$$

Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis for V. In this case we have

$$e_i^2 = e_i e_i = e_i \cdot e_i + e_i \wedge e_i = e_i \cdot e_i = g(e_i, e_i), \tag{10}$$

$$e_{ij} = e_i e_j = e_i \cdot e_j + e_i \wedge e_j = e_i \wedge e_j \quad (i \neq j). \tag{11}$$

Since exterior, Grassmann and Clifford algebras are isomorphic as vector spaces, a general element of $Cl(V, g)$ is of the form

$$A = a_0 + a^i e_i + a^{ij} e_{ij} + \cdots + a^{1 \cdots n} e_{1 \cdots n}, \tag{12}$$

where, in particular, $e_{1 \cdots n} = e_1 \cdots e_n = e_1 \wedge \cdots \wedge e_n = \tau$ is the volume element of V. We observe that Clifford algebra is Z_2-graded algebra. In fact, let Cl_+ (Cl_-) denote the set of elements of Cl with even (odd) grade; we have $Cl_+ Cl_+ \subset Cl_+$, $Cl_+ Cl_- \subset Cl_-$, $Cl_- Cl_+ \subset Cl_-$, $Cl_- Cl_- \subset Cl_+$. This fact shows that Cl_+ is a sub-algebra of Cl, called the even sub-algebra.

The great advantage of Clifford algebra over Grassmann algebra follows from the fact that the Clifford product contains more information than Grassmann product. Moreover, under the Clifford product it is possible to "divide" multivectors. In fact, define the norm $|A|$ of a multivector A as

$$|A|^2 = \langle \tilde{A} A \rangle_0. \tag{13}$$

If $\tilde{A} A = |A|^2 \neq 0$ we define $A^{-1} = \tilde{A} |A|^{-2}$; in fact $A^{-1} A = A A^{-1} = |A|^{-2} \tilde{A} A = 1$. Another advantage is that Clifford algebras are isomorphic to matrix algebras (or direct sums) over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ (we shall see examples below) [Fi90].

Finally, we say that an algebraic element e is an idempotent if $e^2 = e$; it is called primitive if it cannot be written as the sum of two mutually annihilating idempotents (that is: $e \neq e' + e''$ with $(e')^2 = e'$, $(e'')^2 = e''$, $e'e'' = e''e' = 0$). The sub-algebra I_E of an algebra A is called a left ideal if given $i \in I_E$ we have $xi \in I_E$, $\forall x \in A$ (similarly for right ideals). An ideal is said to be minimal if it contains only trivial sub-ideals. Now, one can prove that the minimal left ideals of a Clifford algebra $Cl(V, g)$ are of the form $Cl(V, g)e$, where e is a primitive idempotent [Po81]. This is an important result to be used later.

B. Spacetime Algebra

Let $V = \mathbb{R}^{1,3}$ be Minkowski vector space, and choose a basis $\{\Gamma_\mu\}$ ($\mu = 0, 1, 2, 3$) such that $g(\Gamma_\mu, \Gamma_\nu) = \eta_{\mu\nu} = \operatorname{diag}(1, -1, -1, -1)$. The spacetime algebra (STA) [He66] is the Clifford algebra of $(\mathbb{R}^{1,3}, \eta)$, denoted by $\mathbb{R}_{1,3}$. Observe that $\Gamma_0^2 = -\Gamma_1^2 = -\Gamma_2^2 = -\Gamma_3^2 = 1$. A general element of $\mathbb{R}_{1,3}$ is of the form

$$A = a + a^\mu \Gamma_\mu + \frac{a^{\mu\nu}}{2!} \Gamma_{\mu\nu} + \frac{a^{\mu\nu\sigma}}{3!} \Gamma_{\mu\nu\sigma} + a^{0123} \Gamma_5, \tag{14}$$

where $\Gamma_{\mu\nu} = \Gamma_\mu \wedge \Gamma_\nu$ ($\mu \neq \nu$) and $\Gamma_5 = \Gamma_0 \Gamma_1 \Gamma_2 \Gamma_3 = \Gamma_0 \wedge \Gamma_1 \wedge \Gamma_2 \wedge \Gamma_3$ is the volume element. Note that $\Gamma_5^2 = -1$ and $\Gamma_5 \Gamma_\mu = -\Gamma_\mu \Gamma_5$.

The STA is particularly useful in formulating the Lorentz rotations. Let $\mathbb{R}_{1,3}^+$ denote the even sub-algebra of $\mathbb{R}_{1,3}$, and let $\mathcal{N}$ be the norm map, i.e., $\mathcal{N}(L) = |L|^2 = \langle \tilde{L} L \rangle_0$. The double covering of the restricted Lorentz group $SO_+(1, 3)$ is $\operatorname{Spin}_+(1, 3)$ defined as

$$\operatorname{Spin}_+(1, 3) = \{R \in \mathbb{R}_{1,3}^+ \mid \mathcal{N}(R) = 1\}. \tag{15}$$

An arbitrary Lorentz rotation is therefore given by $a \mapsto RaR^{-1} = Ra\tilde{R}$, with $R \in \mathrm{Spin}_+(1,3)$. One can also prove that any $R \in \mathrm{Spin}_+(1,3)$ can be written in the form $R = \pm e^B$ with $B \in \bigwedge^2(\mathbb{R}^{1,3})$, and the choice of the sign can always be the positive one except when $R = -e^B$ with $B^2 = 0$. When B is a timelike bivector ($B^2 > 0$) R describes a boost, while when B is a spacelike bivector ($B^2 < 0$) R describes a spatial rotation.

We said that Clifford algebras are isomorphic to matrix algebras. In the case of $\mathbb{R}_{1,3}$ it is isomorphic to $\mathcal{M}(2, \mathbb{H})$, the algebra of 2×2 matrices over the quaternions. This isomorphism defines representations of STA. One representation is:

$$\Gamma_0 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \Gamma_1 \leftrightarrow \begin{pmatrix} 0 & \boldsymbol{i} \\ \boldsymbol{i} & 0 \end{pmatrix},$$

$$\Gamma_2 \leftrightarrow \begin{pmatrix} 0 & \boldsymbol{j} \\ \boldsymbol{j} & 0 \end{pmatrix}, \Gamma_3 \leftrightarrow \begin{pmatrix} 0 & \boldsymbol{k} \\ \boldsymbol{k} & 0 \end{pmatrix}, \tag{16}$$

where $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1$ and $\boldsymbol{ij} = \boldsymbol{k}$, $\boldsymbol{jk} = \boldsymbol{i}$, $\boldsymbol{ki} = \boldsymbol{j}$.

Finally, consider the idempotent $e = \frac{1}{2}(1 + \Gamma_0)$. $I_{1,3} = \mathbb{R}_{1,3}e$ is a minimal left ideal. One can easily prove that

$$I_{1,3} = \mathbb{R}_{1,3}e = \mathbb{R}_{1,3}^+ e, \tag{17}$$

which is an important result to be used later.

C. Dirac Algebra

Consider the vector space $\mathbb{R}^{4,1}$ and a basis $\{E_a\}$ ($a = 0, 1, 2, 3, 4$) such that $E_1^2 = E_2^2 = E_3^2 = E_4^2 = -E_0^2 = 1$. Let $\mathbb{R}_{4,1}$ be its Clifford algebra and $i = E_0 E_1 E_2 E_3 E_4$ be the volume element. Note that $i^2 = -1$, but differently from STA now we have $E_a i = i E_a$ ($\forall a$), that is, $\mathbb{R} \oplus i\mathbb{R}$ is the center of $\mathbb{R}_{4,1}$. The volume element i plays therefore the role of imaginary unity. Let us define

$$\Gamma_\mu = E_\mu E_4 \quad (\mu = 0, 1, 2, 3). \tag{18}$$

One can easily see from the above map and with i playing the role of imaginary unity that $\mathbb{R}_{4,1}$ is isomorphic to the complexified STA:

$$\mathbb{R}_{4,1} \simeq \mathbb{C} \otimes \mathbb{R}_{1,3}. \tag{19}$$

Moreover, the even subalgebra of $\mathbb{R}_{4,1}$ is isomorphic to $\mathbb{R}_{1,3}$:

$$\mathbb{R}_{4,1}^+ \simeq \mathbb{R}_{1,3}. \tag{20}$$

The algebra $\mathbb{R}_{4,1}$, or equivalently, the complexified STA, is called Dirac algebra. In fact, they are isomorphic to $\mathcal{M}(4, \mathbb{C})$ – the algebra of 4×4 matrices over the complexes. One representation (the standard one) of Γ_μ in eq.(18) is:

$$\Gamma_0 \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \; \Gamma_1 \leftrightarrow \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\Gamma_2 \leftrightarrow \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \; \Gamma_3 \leftrightarrow \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{21}$$

Consider the idempotent $f = \frac{1}{2}(1 + \Gamma_0)\frac{1}{2}(1 + i\Gamma_{12}) = e\frac{1}{2}(1 + i\Gamma_{12})$, where $e = \frac{1}{2}(1 + \Gamma_0)$ is a primitive idempotent of $\mathbb{R}_{1,3}$. Then $I_{4,1} = \mathbb{R}_{4,1}f$ is a minimal left ideal, and one can show that

$$I_{4,1} = \mathbb{R}_{4,1}f \simeq \mathbb{R}_{4,1}^+ f. \tag{22}$$

D. Dirac-Hestenes Spinors

Let us consider a Dirac spinor $|\Psi\rangle \in \mathbb{C}^4$. There is an obvious isomorphism between $\mathbb{C}^4$ and minimal left ideals of $\mathcal{M}(4,\mathbb{C})$, given by

$$\mathbb{C}^4 \ni |\Psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \leftrightarrow \begin{pmatrix} \psi_1 & 0 & 0 & 0 \\ \psi_2 & 0 & 0 & 0 \\ \psi_3 & 0 & 0 & 0 \\ \psi_4 & 0 & 0 & 0 \end{pmatrix} = \Psi \in \text{ideal of } \mathcal{M}(4,\mathbb{C}). \tag{23}$$

One can, of course, work with Ψ instead of $|\Psi\rangle$, and since $\mathcal{M}(4,\mathbb{C})$ is a representation of Dirac algebra $\mathbb{R}_{4,1} \simeq \mathbb{C} \otimes \mathbb{R}_{1,3}$, one can work with the corresponding ideal of Dirac algebra. Note that

$$\begin{pmatrix} \psi_1 & 0 & 0 & 0 \\ \psi_2 & 0 & 0 & 0 \\ \psi_3 & 0 & 0 & 0 \\ \psi_4 & 0 & 0 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} \psi_1 & \cdot & \cdot & \cdot \\ \psi_2 & \cdot & \cdot & \cdot \\ \psi_3 & \cdot & \cdot & \cdot \\ \psi_4 & \cdot & \cdot & \cdot \end{pmatrix}}_{\in \mathcal{M}(4,C)} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{f}, \tag{24}$$

where f is a matrix representation of the idempotent $f = \frac{1}{2}(1 + \Gamma_0)\frac{1}{2}(1 + i\Gamma_{12})$. One can work therefore with the ideal $I_{4,1} = \mathbb{R}_{4,1}f$ instead of $\mathbb{C}^4$. But the isomorphisms discussed in the preceeding subsections tell us that

$$I_{4,1} = \mathbb{R}_{4,1}f = (\mathbb{C} \otimes \mathbb{R}_{1,3})f \simeq \mathbb{R}_{4,1}^+f \simeq \mathbb{R}_{1,3}f = (\mathbb{R}_{1,3}e)\frac{1}{2}(1 + i\Gamma_{12}). \tag{25}$$

Note that in the last equality we have a minimal left ideal $\mathbb{R}_{1,3}e$ of STA. These equalities show that all informations we obtain from an element of the ideal $I_{4,1}$ can be obtained from an element of the ideal $I_{1,3} = \mathbb{R}_{1,3}e$. Moreover, we have that

$$if = \Gamma_{21}f. \tag{26}$$

These results mean that we can work with the ideal $\mathbb{R}_{1,3}e$ once we identify Γ_{21} as playing in STA the role of the imaginary unity i.

Now, we saw that

$$\mathbb{R}_{1,3}e = \mathbb{R}_{1,3}^+e. \tag{27}$$

What the idempotent makes is "kill" redundant degrees of freedom. Since $\dim\mathbb{R}_{1,3}^+ = 8$ we can work with $\mathbb{R}_{1,3}^+$ instead of $\mathbb{R}_{1,3}e$ (this is not the case for $\mathbb{R}_{1,3}$ or $\mathbb{C} \otimes \mathbb{R}_{1,3}$ since $\dim\mathbb{R}_{1,3} = 16$ and $\dim(\mathbb{C} \otimes \mathbb{R}_{1,3}) = 32$). We established therefore the isomorphism

$$\mathbb{C}^4 \simeq \mathbb{R}_{1,3}^+. \tag{28}$$

The representative of the Dirac spinor $|\Psi\rangle$ in $\mathbb{R}_{4,1}$ is Ψ, which is related to $\psi \in \mathbb{R}_{1,3}^+$ by

$$\Psi = \psi\frac{1}{2}(1 + \Gamma_0)\frac{1}{2}(1 + i\Gamma_{12}). \tag{29}$$

Such ψ will be called Dirac-Hestenes spinor. Its (standard) matrix representation is:

$$\psi \leftrightarrow \begin{pmatrix} \psi_1 & -\psi_2^* & \psi_3 & \psi_4^* \\ \psi_2 & \psi_1^* & \psi_4 & -\psi_3^* \\ \psi_3 & \psi_4^* & \psi_1 & -\psi_2^* \\ \psi_4 & -\psi_3^* & \psi_2 & \psi_1^* \end{pmatrix}. \tag{30}$$

Now, $\psi \in \mathbb{R}_{1,3}^+$, so that its general form is

$$\psi = a + a_{01}\Gamma_{01} + a_{02}\Gamma_{02} + a_{03}\Gamma_{03}$$
$$+a_{12}\Gamma_{12} + a_{13}\Gamma_{13} + a_{23}\Gamma_{23} + a_{0123}\Gamma_5. \tag{31}$$

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From the representation (16) we have

$$\Gamma_{12} = \boldsymbol{k}1, \quad \Gamma_{31} = \boldsymbol{j}1, \quad \Gamma_{23} = \boldsymbol{i}1,$$
$$\Gamma_{01} = -\Gamma_5 \boldsymbol{i}1, \quad \Gamma_{02} = -\Gamma_5 \boldsymbol{j}1, \quad \Gamma_{03} = -\Gamma_5 \boldsymbol{k}1, \tag{32}$$

where 1 is the identity 2×2 matrix and Γ_5 commutes with the quaternion units $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$. ψ can therefore be represented by a biquaternion number, that is, by $A + \Gamma_5 B$ with A and B quaternions, or, since Γ_5 commutes with $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ and $\Gamma_5^2 = -1$, by a quaternion over the complexes.

One important comment that must be made here concerns the transformation law of a Dirac-Hestenes spinor. In order to obtain a correct transformation law under the action of the Lorentz group, we must be more precise [RS95] and say that a Dirac-Hestenes spinor is an element of the quotient set $I\!\!R_{1,3}^+/\mathcal{R}$ such that given the orthonormal basis Σ and $\dot{\Sigma}$ of $I\!\!R^{1,3} \subset I\!\!R_{1,3}$, $\psi_\Sigma \in I\!\!R_{1,3}^+$, $\psi_{\dot{\Sigma}} \in I\!\!R_{1,3}^+$, then $\psi_\Sigma \sim \psi_{\dot{\Sigma}}$ (mod $\mathcal{R}$) if and only if $\psi_{\dot{\Sigma}} = \psi_\Sigma U^{-1}$ with $\dot{\Sigma} = \mathcal{L}(\Sigma) = U\Sigma U^{-1}$, $U \in \mathrm{Spin}_+(1,3)$, $\mathcal{L} \in \mathrm{SO}_+(1,3)$ and $\mathcal{H}(U) = \mathcal{L}$ where $\mathcal{H}$ is the double covering $\mathcal{H}: \mathrm{Spin}_+(1,3) \to \mathrm{SO}_+(1,3)$. We say that ψ_Σ is the representative of the Dirac-Hestenes spinor in the orthonormal basis Σ, and when no confusion arises we write only ψ instead of ψ_Σ.

Suppose now that ψ is non-singular, i.e., $\psi\tilde{\psi} \neq 0$. Since $\psi \in I\!\!R_{1,3}^+$ we have

$$\psi\tilde{\psi} = \sigma + \Gamma_5\omega, \tag{33}$$

where σ and ω are scalars. Define quantities $\rho = \sqrt{\sigma^2 + \omega^2}$ and $\tan\beta = \omega/\sigma$. Then ψ can be written as

$$\psi = \sqrt{\rho}e^{\Gamma_5\beta/2}R, \tag{34}$$

where $R \in \mathrm{Spin}_+(1,3)$, $\rho \in I\!\!R_+$ and $0 \leq \beta < 2\pi$. This is the canonical decomposition of Dirac-Hestenes spinors in terms of a Lorentz rotation R, a dilation ρ and a duality transformation by an angle β (called Yvon-Takabayasi angle). This is a remarkable result since it generalizes the polar decomposition of a complex number for the case of a biquaternion, and has a clear geometrical interpretation.

Finally, let us write Dirac equation in terms of ψ. Let (M, g, D) be Minkowski spacetime, where (M, g) is a four dimensional time oriented and space oriented Lorentzian manifold with g a Lotentzain metric with signature $(1, 3)$ and D is its Levi-Civita connection. The tangent space at $x \in M$ is $T_xM \simeq I\!\!R^{1,3}$ and the tangent bundle is $T(M) = \cup_{x \in M} T_xM$. We represent by $\bigwedge(M) = \cup_{x \in M} \bigwedge(T_xM)$ the Cartan bundle of multivectors and by $\mathcal{C}\ell(M) = \cup_{x \in M} \mathcal{C}\ell(T_xM)$ the Clifford bundle of multivectors, where $\mathcal{C}\ell(T_xM) \simeq I\!\!R_{1,3}$. Dirac-Hestenes spinor fields are sections of the so called Spin-Clifford bundle $\mathcal{C}\ell_{\mathrm{Spin}_+(1,3)}(M) \simeq \mathcal{C}\ell(M)/\mathcal{R}$, where $\mathcal{R}$ is the equivalence relation defined above. This means that given an orthonormal basis $\Sigma = \{\gamma_\mu\}$, where $\gamma_\mu \in \sec T(M) \in \sec\mathcal{C}\ell(M)$, the representative of a Dirac-Hestenes spinor field in the basis Σ is an even section of $\mathcal{C}\ell(M)$. Note that for each $x \in M$ we have $\gamma_\mu \simeq \Gamma_\mu$. Introducing $\{\gamma^\mu\}$ as the reciprocal basis, i.e., $\gamma^\mu \cdot \gamma_\nu = \delta_\nu^\mu$. We have $\gamma^\mu \simeq \Gamma^\mu$ for each $x \in M$. With the above remarks, and using some previous results, namely $|\Psi\rangle = \psi f$, $\gamma_{21}f = if$ and $\gamma_0 f = f$, the (free) Dirac equation $i\partial |\Psi(x)\rangle = (mc/\hbar)|\Psi(x)\rangle$, where $|\Psi(x)\rangle$ is a Dirac spinor field [Ch68], is written in STA as [RO90]

$$\partial\psi\gamma_{21} = \frac{mc}{\hbar}\psi\gamma_0, \tag{35}$$

which we call Dirac-Hestenes equation. ∂ is the Dirac operator, which in an orthonormal coordinate basis is $\partial = \gamma^\mu \frac{\partial}{\partial x^\mu}$.

It is interesting to observe that γ_{21} appears on the right of ψ in eq.(35). The reason is the equation $\gamma_{21}f = if$, from which we have $\psi\gamma_{21}f = \psi if = i\psi f = i|\psi\rangle$. If we represent ψ by means of quaternions, a quaternion unit has to be introduced multiplying ψ on the right. This is the reason why de Leo and Rotelli [LR94] had to introduced left and right acting elements in their quaternionic version of quantum mechanics.

III. SPINORIAL REPRESENTATION OF MAXWELL EQUATIONS

The spinorial representation of Maxwell equations we shall give in this section is based on the following theorem:

Theorem: Any electromagnetic field $F \in \sec\bigwedge^2(M) \in \sec\mathcal{C}\ell(M)$ can be written in the form

$$F = \psi\gamma_{21}\tilde{\psi}, \tag{36}$$

where ψ is a Dirac-Hestenes spinor field.

The proof of this theorem can be divided in three steps: (i) $F^2 \neq 0$; (ii) $F^2 = 0$, $F \neq 0$; (ii) $F^2 = 0$, $F = 0$. In the first case the proof is based on a theorem by Rainich [Ra25], and reconsidered by Misner and Wheeler [MW57].

Theorem (Rainich-Misner-Wheeler): Let an extremal field be an electromagnetic field for which the electric [magnetic] field vanishes and the magnetic [electric] field is parallel to a given spatial direction. Then, at any point of spacetime, any non-null ($F^2 \neq 0$) electromagnetic field F can be transformed in an extremal field by means of a Lorentz transformation and a duality transformation.

An easy proof of the theorem of Rainich-Misner-Wheeler can be found in [VR93]. Now consider a Lorentz transformation $F \mapsto F' = LF\tilde{L}$, and a duality transformation $F' \mapsto F'' = e^{\gamma_5 \alpha} F'$. According to the theorem of RMW the electromagnetic field F'' is extremal; let us suppose it of magnetic type along the $\vec{k}$ direction, that is: $F'' = \rho \gamma_{21}$, where ρ is the extremal field intensity. We have therefore that $\rho \gamma_{21} = e^{\gamma_5 \alpha} LF\tilde{L}$. Let us define $R = \tilde{L}$ and $\beta = -\alpha$; then we have that

$$F = \psi \gamma_{21} \tilde{\psi}, \tag{37}$$

where

$$\psi = \sqrt{\rho} e^{\gamma_5 \beta/2} R, \tag{38}$$

which we recognize as the canonical decomposition of Dirac-Hestenes spinor.

In order to prove our theorem for case (ii) we observe that since $F^2 = 0$ we have $\vec{E} \cdot \vec{H} = 0$ and $\vec{E}^2 = \vec{H}^2$; we can make therefore a spatial rotation $\mathcal{R}$ such that $E_1' = H_1' = 0$ and $H_3' = \pm E_2' = \eta_1$ and $H_2' = \pm E_3' = \eta_2$; then for $F' = (1/2)(F')^{\mu\nu} \gamma_{\mu\nu}$ we have $F' = (\eta_1 + \gamma_5 \eta_2)(1/2)(1 \pm \gamma_{01}) \gamma_{21}$. If we take $R = \tilde{\mathcal{R}}$ we have for $F = RF'\tilde{R}$ that $F = (\eta_1 + \gamma_5 \eta_2) R (1/2)(1 \pm \gamma_{01}) \gamma_{21} \tilde{R}$. Remember that $(1/2)(1 \pm \gamma_{01})$ is an idempotent; defining $\eta_1 = \eta \cos \varphi$ and $\eta_2 = \eta \sin \varphi$ it follows that

$$F = \psi_M \gamma_{21} \tilde{\psi}_M, \tag{39}$$

where

$$\psi_M = \sqrt{\eta} e^{\gamma_5 \varphi/2} R \frac{1}{2}(1 \pm \gamma_{01}) = \psi \frac{1}{2}(1 \pm \gamma_{01}), \tag{40}$$

which proves our assertion in this case. ψ_M is a particular type of Dirac-Hestenes spinor known as Majorana spinor [Lo93].

Now for case (iii) ($F = 0$) we note that $\psi \gamma_{21} \tilde{\psi} = -\psi \gamma_{21} \tilde{\psi} = \gamma_5 \psi \gamma_{21} \tilde{\psi} \gamma_5 = \gamma_5 \psi \gamma_{21} \gamma_{21} \gamma_{12} \tilde{\psi} \gamma_5$ is satisfied for $\psi = \pm \gamma_5 \psi \gamma_{21}$. It follows therefore that

$$F = \psi_W \gamma_{21} \tilde{\psi}_W = 0, \tag{41}$$

where

$$\psi_W = \frac{1}{2}(\psi \pm \gamma_5 \psi \gamma_{21}). \tag{42}$$

This particular kind of Dirac-Hestenes spinor is called a Weyl spinor [Lo93]. We have now proved our theorem. In terms of usual Dirac spinor fields eq.(36) gives

$$F_{\mu\nu} = \langle \Psi | \frac{i}{2}[\gamma_\mu, \gamma_\nu] | \Psi \rangle, \tag{43}$$

where γ_μ are Dirac matrices, that is, the matrix representation of vectors γ_μ as in eq.(21), $|\Psi\rangle$ is the Dirac spinor field and $\langle \Psi |$ its Dirac adjoint spinor field [Ch68].

Remark 1: The Faraday bivector F is given by a quadractic expression in terms of the Dirac-Hestenes spinor according to eq.(36). In this way we speak of the Faraday bivector as the "Dirac square" of the Dirac-Hestenes spinor, since eq.(36) is a particular kind of square involving the bivector γ_{21} which plays the role of an unity, in this case an unity of extremal field. Similarly, we speak of the Dirac-Hestenes spinor as the "Dirac square root" of the Faraday bivector.

Remark 2: It remains, of course, the question of the constants in eq.(36) since the units of F are charge $\times$ (length)$^{-2}$ and the units of ψ are (length)$^{-3/2}$. We have now to introduce two postulates: firstly, we suppose that there is a

natural unit F_0 of electromagnetic field intensity; secondly, we suppose that there is a natural unit e_0 of electric charge. In this way we have that

$$F = \left(\frac{e_0^3}{F_0}\right)^{\frac{1}{2}} \psi\gamma_{21}\tilde{\psi}. \tag{44}$$

Note that $(e_0/F_0)^{1/2}$ has unit of length. One of the above postulates can be replaced by the one that there is a natural unit L_0 of length, which gives $F_0 = e_0/L_0^2$ and

$$F = e_0 L_0 \psi\gamma_{21}\tilde{\gamma}. \tag{45}$$

In terms of some physical constants, a combination of constants we need can be

$$F = \frac{e\hbar}{2mc}\psi\gamma_{21}\tilde{\psi}, \tag{46}$$

which gives correct units for F and ψ. In this expression the symbols have their usual meaning, that is: e is the elementary electric charge, $\hbar$ is the Planck constant, m is the electron mass and c the velocity of light in vacuum. In what follows we will work, to simplify the notation, with eq.(36) instead of eq.(46).

The idea now is to use $F = \psi\gamma_{21}\tilde{\psi}$ in Maxwell equations and obtain from it an equivalent equation for ψ. Maxwell equations using STA are written as an unique equation, namely [He66,Ma90]

$$\partial F = \mathcal{J}, \tag{47}$$

where $\partial = \gamma^\mu\partial_\mu$ is the Dirac operator and $\mathcal{J}$ is the electromagnetic current (an electric current J_e plus a magnetic monopolar current $\gamma_5 J_m$ in the general case). If we use eq.(36) in Maxwell equation (47) we obtain

$$\partial(\psi\gamma_{21}\tilde{\psi}) = \gamma^\mu\partial_\mu(\psi\gamma_{21}\tilde{\psi}) = \gamma^\mu(\partial_\mu\psi\gamma_{21}\tilde{\psi} + \psi\gamma_{21}\partial_\mu\tilde{\psi}) = \mathcal{J}. \tag{48}$$

But $\psi\gamma_{21}\partial\tilde{\psi} = -(\partial_\mu\psi\gamma_{21}\tilde{\psi})\tilde{}$, and since reversion does not change the sign of scalars and of pseudo-scalars (4-vectors), we have that

$$2\gamma^\mu\langle\partial_\mu\psi\gamma_{21}\tilde{\psi}\rangle_2 = \mathcal{J}. \tag{49}$$

There is a more convenient way of rewriting the above equation. Note that

$$\gamma^\mu\langle\partial_\mu\psi\gamma_{21}\tilde{\psi}\rangle_2 = \partial\psi\gamma_{21}\tilde{\psi} - \gamma^\mu\langle\partial_\mu\psi\gamma_{21}\tilde{\psi}\rangle_0 - \gamma^\mu\langle\partial_\mu\psi\gamma_{21}\tilde{\psi}\rangle_4, \tag{50}$$

and if we define the vectors

$$j = \gamma^\mu\langle\partial_\mu\psi\gamma_{21}\tilde{\psi}\rangle_0, \tag{51}$$

$$g = \gamma^\mu\langle\partial_\mu\psi\gamma_5\gamma_{21}\tilde{\psi}\rangle_0, \tag{52}$$

we can rewrite eq.(49) as

$$\partial\psi\gamma_{21}\tilde{\psi} = \left[\frac{1}{2}\mathcal{J} + (j + \gamma_5 g)\right]. \tag{53}$$

If correct units have been used, that is, if we used eq.(46), then instead of $(1/2)\mathcal{J}$ we would obtained $(mc/e\hbar)\mathcal{J}$. Eq.(53) is the spinorial representation of Maxwell equations we were looking for. In the case where ψ is non-singular (which corresponds to non-null electromagnetic fields) we have

$$\partial\psi\gamma_{21} = \frac{e^{\gamma_5\beta}}{\rho}\left[\frac{1}{2}\mathcal{J} + (j + \gamma_5 g)\right]\psi. \tag{54}$$

Eq.(54) has been proved [VR93] to be equivalent to the spinorial representation of Maxwell equations obtained originally by Campolattaro [Ca90] in terms of the usual covariant Dirac spinor.

IV. IS THERE ANY RELATIONSHIP BETWEEN MAXWELL AND DIRAC EQUATIONS?

The spinorial eq.(54) that represents Maxwell ones, as written in that form, does not appear to have any relationship with Dirac equation (35). However, we shall make some modifications on it in such a way to put it in a form that suggests a very interesting and intriguing relationship between them, and consequently between electromagnetism and quantum mechanics.

Since ψ is supposed to be non-singular (F non-null) we can use the canonical decomposition (34) of ψ and write

$$\partial_\mu \psi = \frac{1}{2} \left(\partial_\mu \ln \rho + \gamma_5 \partial_\mu \beta + \Omega_\mu \right) \psi, \tag{55}$$

where we defined

$$\Omega_\mu = 2(\partial_\mu R)\tilde{R}. \tag{56}$$

Using this expression for $\partial_\mu \psi$ into the definitions of the vectors j and g (eqs.(51,52)) we obtain that

$$j = \gamma^\mu (\Omega_\mu \cdot S)\rho \cos\beta + \gamma_\mu [\Omega_\mu \cdot (\gamma_5 S)]\rho \sin\beta, \tag{57}$$

$$g = \gamma^\mu [(\Omega_\mu \cdot (\gamma_5 S)]\rho \cos\beta - \gamma_\mu (\Omega_\mu \cdot S)\rho \sin\beta, \tag{58}$$

where we defined the bivector S by

$$S = \frac{1}{2}\psi \gamma_{21} \psi^{-1} = \frac{1}{2} R \gamma_{21} \tilde{R}. \tag{59}$$

A more convenient expression can be written. Let v be given by $\rho v = J = \psi \gamma_0 \tilde{\psi}$, and $v_\mu = v \cdot \gamma_\mu$. Define the bivector $\Omega = v^\mu \Omega_\mu$ and the scalars Λ and K by

$$\Lambda = \Omega \cdot S, \tag{60}$$

$$K = \Omega \cdot (\gamma_5 S). \tag{61}$$

Using these definitions we have that

$$\Omega_\mu \cdot S = \Lambda v_\mu, \tag{62}$$

$$\Omega_\mu \cdot (\gamma_5 S) = K v_\mu, \tag{63}$$

and for the vectors j and g:

$$j = \Lambda v\rho \cos\beta + K v\rho \sin\beta = \lambda\rho v, \tag{64}$$

$$g = K v\rho \cos\beta - \Lambda v\rho \sin\beta = \kappa\rho v, \tag{65}$$

where we defined

$$\lambda = \Lambda \cos\beta + K \sin\beta, \tag{66}$$

$$\kappa = K \cos\beta - \Lambda \sin\beta. \tag{67}$$

The spinorial representation of Maxwell equations are written now as

$$\partial\psi\gamma_{21} = \frac{e^{\gamma_5 \beta}}{2\rho} \mathcal{J}\psi + \lambda\psi\gamma_0 + \gamma_5\kappa\psi\gamma_0. \tag{68}$$

If $\mathcal{J} = 0$ (free case) we have that

$$\partial\psi\gamma_{21} = \lambda\psi\gamma_0 + \gamma_5\kappa\psi\gamma_0, \tag{69}$$

which is very similar to Dirac equation (35).

In order to go a step further into the relationship between those equations, we remember that the electromagnetic field has six degrees of freedom, while a Dirac-Hestenes spinor field has eight degrees of freedom; we are free therefore to impose two constraints on ψ if it is to represent an electromagnetic field [Ca90]. We choose these two constraints as

$$\partial\cdot j = 0 \quad\text{and}\quad \partial\cdot g = 0. \tag{70}$$

Using eqs.(64,65) these two constraints become

$$\partial\cdot j = \rho\dot{\lambda} + \lambda\partial\cdot J = 0, \tag{71}$$

$$\partial\cdot g = \rho\dot{\kappa} + \kappa\partial\cdot J = 0, \tag{72}$$

where $J = \rho v$ and $\dot{\lambda} = (v\cdot\partial)\lambda$, $\dot{\kappa} = (v\cdot\partial)\kappa$. These conditions imply that

$$\kappa\dot{\lambda} = \lambda\dot{\kappa}, \tag{73}$$

which gives $(\lambda \neq 0)$:

$$\frac{\kappa}{\lambda} = \text{const} = -\tan\beta_0, \tag{74}$$

or from eqs.(66,67):

$$\frac{K}{\Lambda} = \tan(\beta - \beta_0). \tag{75}$$

Now we observe that β is the angle of the duality rotation from F to $F' = e^{\gamma_5\beta}F$. If we perform another duality rotation by β_0 we have $F \mapsto e^{\gamma_5(\beta+\beta_0)}F$, and for the Yvon-Takabayasi angle $\beta \mapsto \beta + \beta_0$. If we work therefore with an electromagnetic field duality rotated by an additional angle β_0, the above relationship becomes

$$\frac{K}{\Lambda} = \tan\beta. \tag{76}$$

This is, of course, just a way to say that we can choose the constant β_0 in eq.(74) to be zero. Now, this expression gives

$$\lambda = \Lambda\cos\beta + \Lambda\tan\beta\sin\beta = \frac{\Lambda}{\cos\beta}, \tag{77}$$

$$\kappa = \Lambda\tan\beta\cos\beta - \Lambda\sin\beta = 0, \tag{78}$$

and the spinorial representation (69) of the free Maxwell equations becomes

$$\partial\psi\gamma_{21} = \lambda\psi\gamma_0. \tag{79}$$

Note that λ is such that

$$\rho\dot{\lambda} = -\lambda\partial\cdot J. \tag{80}$$

The current $J = \psi\gamma_0\tilde{\psi}$ is not conserved unless λ is constant. If we suppose also that

$$\partial\cdot J = 0 \tag{81}$$

we must have

$$\lambda = \text{const.} \tag{82}$$

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Now, throughout these calculations we have assumed $\hbar = c = 1$. We observe that in eq.(79) λ has the units of $(\text{length})^{-1}$, and if we introduce the constants $\hbar$ and c we have to introduce another constant with unit of mass. If we denote this constant by m such that

$$\lambda = \frac{mc}{\hbar}, \tag{83}$$

then eq.(79) assumes a form which is identical to Dirac equation:

$$\partial \psi \gamma_{21} = \frac{mc}{\hbar} \psi \gamma_0. \tag{84}$$

It is true that we didn't proved that eq.(84) is really Dirac equation since the constant m has to be identified in this case with the electron's mass. However, we shall make some remarks concerning this identification which are very interesting and intriguing. First, if in analogy to eq.(83) we write

$$\Lambda = \frac{Mc}{\hbar}, \tag{85}$$

then eq.(77) reads

$$m = \frac{M}{\cos \beta}, \tag{86}$$

or

$$M = m \cos \beta = \frac{m}{\sqrt{1 + \omega^2 / \sigma^2}}, \tag{87}$$

where σ and ω are the invariants of Dirac theory and given by eq.(33). If m is constant, the above expression defines a variable mass M.

On the other hand, de Broglie introduced in his interpretation of quantum mechanics a variable mass M related to the constant one m by de Broglie-Vigier formula [Br67] $M = m\sqrt{1 + \omega^2 / \sigma^2}$, which is very similar the our one. However, the difference in these formulas are unimportant for the free case where we have for the plane wave solutions that [He90] $\cos \beta = \pm 1$. Another interesting fact comes from eq.(60). If we write $\psi = \psi_0 \exp(\gamma_{21} \omega t)$, where ψ_0 is a constant spinor (which is the case again for plane waves), then eq.(60) gives $\Lambda = \omega$, or, introducing the constants $\hbar$ and c:

$$Mc^2 = \hbar \omega. \tag{88}$$

The variable mass M now appears to be related to energy of some "internal" vibration, and eq.(88) is just another formula of de Broglie [Br70], who suggested that mass is related to the frequency of an internal clock supposed associated to a particle.

We can better understand the meaning of eq.(60) after some additional manipulations. We have

$$\Lambda = \Omega \cdot S = v^\mu \langle \Omega_\mu \frac{1}{2} \psi \gamma_{21} \psi^{-1} \rangle_0, \tag{89}$$

and using eq.(55):

$$\Lambda = v^\mu \langle \partial_\mu \gamma_{21} \psi^{-1} \rangle_0 - v^\mu (\partial_\mu \ln \rho) \langle \psi \gamma_{21} \psi^{-1} \rangle_0 - v^\mu \partial_\mu \beta \langle \gamma_5 \psi \gamma_{21} \psi^{-1} \rangle_0. \tag{90}$$

The second and third terms on the RHS vanish because they are 2-vectors; then, using $\psi^{-1} = \tilde{\psi}(\psi \tilde{\psi})^{-1}$,

$$\Lambda = v^\mu \langle \partial_\mu \psi \gamma_{21} \gamma_0 \tilde{\psi} (\psi \tilde{\psi})^{-1} \psi \gamma_0 \tilde{\psi} (\psi \tilde{\psi})^{-1} \rangle_0 =$$
$$= v^\mu \langle \partial_\mu \psi \gamma_{21} \gamma_0 \tilde{\psi} \frac{e^{-\gamma_5 \beta}}{\rho} \rho v \frac{e^{-\gamma_5 \beta}}{\rho} \rangle_0 = v^\mu \frac{1}{\rho} v^\nu \langle \partial_\mu \psi \gamma_{21} \gamma_0 \tilde{\psi} \gamma_\nu \rangle_0. \tag{91}$$

But one can show [He90] that the energy-momentum tensor in Dirac theory (Tetrode tensor) in terms of Dirac-Hestenes spinor is given by

$$T_{\mu\nu} = \langle \partial_\mu \psi \gamma_{21} \gamma_0 \tilde{\psi} \gamma_\nu \rangle_0. \tag{92}$$

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Since

$$v^\nu T_{\mu\nu} = \rho p_\mu \tag{93}$$

it follows, with correct units, the well-known equation

$$Mc = v \cdot p. \tag{94}$$

We remember that for plane waves $\cos\beta = \pm 1$, which implies from eq.(87) that $p = \pm mcv$, with the plus sign corresponding to the positive energy solution and the minus one to the negative energy solution, and which enable us to look as Feynman and Stueckelberg to electron and positron as particles with opposite momenta.

V. OTHER POSSIBLE INTERPRETATIONS

We can go on in our analysis and look to this problem in a different way. Consider Dirac equation, now with an external electromagnetic potential (introduced only for convenience):

$$\partial\phi\gamma_{21} = m\phi\gamma_0 + eA\phi. \tag{95}$$

Let us write the Dirac-Hestenes spinor field ϕ in the following form

$$\phi = \phi_0 e^{-\gamma_{21}\chi}, \tag{96}$$

which is always possible, with $\phi_0 = \sqrt{\rho}e^{\gamma_5\beta/2}R_0$. If we use this expression in Dirac equation we obtain

$$\partial\phi_0\gamma_{21} + \partial\chi\phi_0 = m\phi_0\gamma_0 + eA\phi_0. \tag{97}$$

Since $J = \rho v = \phi\gamma_0\tilde{\phi} = \phi_0\gamma_0\tilde{\phi}_0$, we can also write

$$\partial\phi_0\gamma_{21}\phi_0^{-1} + \partial\chi = me^{\gamma_5\beta}v + eA. \tag{98}$$

This equation has vector and pseudo-vector (3-vector) parts, which are

$$\langle\partial\phi_0\gamma_{21}\phi_0^{-1}\rangle_1 + (\partial\chi - eA) = m\cos\beta v, \tag{99}$$

$$\langle\partial\phi_0\gamma_{21}\phi_0^{-1}\rangle_3 = \gamma_5 m\sin\beta v. \tag{100}$$

Eq.(99) looks as a factorized Hamilton-Jacobi equation with χ playing the role of the action function. Indeed, consider the free case: $A = 0$. In this case we have plane waves solutions for which $\partial\phi_0 = 0$ and $\cos\beta = \pm 1$, and eq.(99) reduces to $\partial\chi = \pm mv$, which squares to $(\partial\chi)^2 = m^2$, which is the relativistic Hamilton-Jacobi equation. When we have interactions, Hamilton-Jacobi equation is modified according to $\partial\chi \mapsto \partial\chi - eA$ (which is just the term in parenthesis on the LHS of eq.(99)) and the factorized Hamilton-Jacobi equation would assumes the form $\partial\chi - eA = \pm mv$. Now, a comparison with eq.(99) shows that the mass term in that factorized Hamilton-Jacobi equation is

$$M = m\cos\beta, \tag{101}$$

which is just our variable mass given by eq.(87). On the other hand, if mass appears as being variable, there must be something that causes it to be so, and this seems to be the first term on the LHS. The natural interpretation of this first term is a quantum electromagnetic potential A_Q:

$$eA_Q = \langle\partial\phi_0\gamma_{21}\phi_0^{-1}\rangle_1. \tag{102}$$

This interpretation looks very nice, but it remains the problem of interpreting the other eq.(100). A possible one consists in a quantum electromagnetic pseudo-potential $\langle\partial\phi_0\gamma_{21}\phi_0^{-1}\rangle_3$, but it is not clear what underlies it in the context of magnetic monopoles (where one can use two electromagnetic potentials – a vector and a pseudo-vector ones – instead of the Dirac string [Ma90]). But it is interesting to observe that in the Schrödinger limit where ϕ_0 is a real scalar the term $(\partial\phi_0)\phi_0^{-1}$, when ϕ_0 has no time dependence, is just the factorization of the term $\nabla^2\phi_0 + |\nabla\phi_0/\phi_0|^2 \phi_0$ (remember that $\partial^2 = \Box = \partial_0^2 - \nabla^2$), which is introduced as nonlinearity in Schrödinger equation in order to have an equation admitting quantum diffusion currents [DG92].

We can also speculate another way of interpreting eqs.(99, 100). We saw in the preceeding section that one can associate a spinor field ψ to an electromagnetic field F by means of $F = \psi \gamma_{21} \tilde{\psi}$. Our idea now is to suppose ϕ_0 associated to a "quantum" electromagnetic field $F = \phi_0 \gamma_{21} \tilde{\phi}_0$. We shall suppose F to satisfies the free Maxwell equations, and in this way ϕ_0 satisfies eq.(54) with $\mathcal{J} = 0$. On the other hand, in this section we imposed two constrains on the spinor field ψ since it has two additional degrees of freedom for representing an electromagnetic field. Now, however, the spinor ϕ_0 has seven degrees of freedom (eq.(96)), and only one constrain has to be imposed. This constrain, however, cannot be arbitrary; since $\phi \gamma_0 \tilde{\phi} = \phi_0 \gamma_0 \tilde{\phi}_0 = \rho v = J$ and $\partial \cdot J = 0$, this constrain must be

$$\partial \cdot (\phi_0 \gamma_0 \tilde{\phi}_0) = \partial \cdot (\rho v) = 0. \tag{103}$$

Now, eq.(54) with $\mathcal{J} = 0$ is (with ψ replaced by ϕ_0, which we indicate by the subscript):

$$\partial \phi_0 \gamma_{21} \phi_0^{-1} = \frac{e^{\gamma_5 \beta}}{\rho} (j_0 + \gamma_5 g_0). \tag{104}$$

Using eqs.(60,61) we have eqs.(64,65) for j_0 and g_0, which in the above equation gives

$$\partial \phi_0 \gamma_{21} \phi_0^{-1} = e^{\gamma_5 \beta} (\lambda_0 + \gamma_5 \kappa_0) v = (\Lambda_0 + \gamma_5 K_0) v, \tag{105}$$

and then

$$\langle \partial \phi_0 \gamma_{21} \phi_0^{-1} \rangle_1 = \Lambda_0 v, \tag{106}$$

$$\langle \partial \phi_0 \gamma_{21} \phi_0^{-1} \rangle_3 = \gamma_5 K_0 v. \tag{107}$$

Using these expressions in eqs.(99,100) we have ($\hbar = c = 1$):

$$\Lambda_0 v + \partial \chi - eA = m \cos \beta v, \tag{108}$$

$$K_0 v = m \sin \beta v. \tag{109}$$

Finally, using the definitions of Λ_0 and K_0 we have

$$\gamma^\mu (\Omega_\mu^{(0)} \cdot S) + \partial \chi - eA = m \cos \beta v, \tag{110}$$

$$\gamma^\mu (\Omega_\mu^{(0)} \cdot (\gamma_5 S)) = m \sin \beta v, \tag{111}$$

where by the superscript (0) we indicate that $\Omega_\mu^{(0)}$ is to be calculated using R_0. Since $R = R_0 \exp(-\gamma_{21} \chi)$ we have

$$\gamma^\mu (\Omega_\mu \cdot S) = \gamma^\mu (\Omega_\mu^{(0)} \cdot S) + \partial \chi, \tag{112}$$

$$\gamma^\mu (\Omega_\mu \cdot (\gamma_5 S)) = \gamma^\mu (\Omega_\mu^{(0)} \cdot (\gamma_5 S)), \tag{113}$$

and we can write that

$$m \cos \beta = \Omega \cdot S - e(A \cdot v), \tag{114}$$

$$m \sin \beta = \Omega \cdot (\gamma_5 S), \tag{115}$$

which in the free case reduce to eqs.(60,61) once we identify Λ with $m \cos \beta = M$ (as we already have done!) and K with $m \sin \beta$ (which is again the case due to eq.(76)!). This shows the consistency of this interpretation.

Finally, we observe that an interesting interpretation for $m \cos \beta$ and $m \sin \beta$ in eqs.(99,100) and eqs.(114,115) would be that of longitudinal and transversal masses, respectively. Further possible interpretations along this line will be postponed for another occasion.

VI. CONCLUSIONS

In this paper we obtained a spinorial representation of Maxwell equations and showed how it can be put in a form which is identical to Dirac equation in the free case. We also discussed certain conditions under which that equation can be really interpreted as Dirac equation. These conditions are related to the interpretation of the mass term, and we developed possible interpretations along some of de Broglie ideas. The natural unit of mass, or the natural unit of length, appears in our approach due to the theorem we proved which gives eq.(36) relating electromagnetic and spinor fields.

A problem to be considered in another occasion is the case when we have sources. In fact, the above relationship between Maxwell and Dirac equations emerged from the free Maxwell equations. However, we have a current $e\psi\gamma_0\tilde{\psi}$ which is a source of a self-field; that is, we have a feedback process. Studies in this case can be found in [Ca90].

We also speculate on an alternative interpretation by splitting the Dirac-Hestenes spinor in a phase given by the action function and a spinor to which we associate a "quantum" electromagnetic field according to a theorem which we proved. This scheme is consistent once that "quantum" electromagnetic field satisfies the free Maxwell equations.

A possible and interesting generalization of our scheme consists in introducing sources for the "quantum" electromagnetic field. An interesting candidate for the current would be a self electromagnetic potential. Indeed, interesting results can be obtained (see, for example, [RV93] and references therein) when the electromagnetic potential plays the role of current in Maxwell equations.

To end we must observe that our construction of the equivalence between Mawell and Dirac equations depends on the existence of non trivial solutions of the free Maxwell equations $\partial F = 0$ such that $F^2 \neq 0$. We proved in [RV95] that non trivial solutions in these conditions indeed exist. They correspond to subluminal and superluminal solutions of Maxwell equations in vacuum!

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